THE HOPF–LAX FORMULA IN CARNOT GROUPS

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Abstract

Let us consider the Heisenberg group $I\!\!H$, the simplest model of Carnot group: in this situation, the horizontal gradient of a function $v : I\!\!H(\sim \mathbb{R}^3) \to \mathbb{R}$ at the point $x = (x_1, x_2, x_3)$ is defined by

$$\mathbb{X}v(x_1, x_2, x_3) = \Big(v_{x_1}(x) - \frac{x_2}{2} v_{x_3}(x),\ v_{x_2}(x) + \frac{x_1}{2} v_{x_3}(x) \Big).$$

We are interested in the problem

$$\text{(1)} \qquad \begin{cases} H\Big(\mathbb{X}u(x,t)\Big) + u_t(x,t) = 0, & \forall (x,t) \in I\!\!H \times (0, \infty) \\ u(x,0) = g(x), & \forall x \in I\!\!H \end{cases}$$

where the Hamiltonian $H : \mathbb{R}^2 \to \mathbb{R}$ is a convex and superlinear function, and the initial date $g : \mathbb{R}^3 \to \mathbb{R}$ is Lipschitz and bounded.

In the particular case $H(\cdot) = \varphi(|\cdot|)$ where $\varphi : [0, \infty) \to \mathbb{R}$ is a convex, increasing, superlinear function, Manfredi and Stroffolini [MS] proved that the unique viscosity solution for (1) is given, for every $x \in I\!\!H$ and $t > 0$, by the Hopf–Lax formula

$$\text{(2)} \qquad u(x,t) = \min_{y \in I\!\!H} \left\{ t\varphi^* \left(\frac{d_{cc}(x,y)}{t} \right) + g(y) \right\}$$

where φ^* is the Legendre–Fenchel transform of the function φ and d_{CC} is the Carnot–Carathéodory distance in $I\!\!H$.

We prove a Hopf–Lax formula for the problem (1) that generalizes (2). Such result is obtained using an optimal control approach similar to the classical case (see, for instance, [BE]). We are able to pass from the Heisenberg situation to a generic Carnot group and to generalize a result due to Dragoni [D].

These recent results are obtained with Z. Balogh and R. Pini.

Key words and phrases. Hamilton–Jacobi equation, Legendre–Fenchel transform, Carnot groups, viscosity solution, Hopf–Lax formula, optimal control problem.

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