The state of the art in shearlet coorbit space theory

One of the most important tasks in modern applied mathematics is the analysis and synthesis of signals. To this end, usually the first step is to decompose the signal with respect to suitable building blocks which are well–suited for the specific application and allow for a fast and efficient extraction of the relevant information. In this context, one particular problem which is currently in the center of interest is the extraction of *directional* information. In recent studies, several approaches have been suggested such as ridgelets, curvelets, contourlets, shearlets and many others. Among all these approaches, the shearlet transform stands out because it is related to group theory, i.e., this transform can be derived from a square-integrable representation $\pi : \mathbb{S} \to \mathcal{U}(L_2(\mathbb{R}^2))$ of a certain group $\mathbb{S}$. This has been clarified in [1] where the underlying group, the *full shearlet group* $\mathbb{S}$, has been established and analyzed. This pure group theoretical approach to shearlets has some important advantages. In particular, it is possible to derive canonical smoothness spaces associated with the shearlet transform. The basic tool to do this is provided by the coorbit space theory derived by Hans Feichtinger and Charly Gröchenig in a series of papers [6, 7, 8, 9]. Under certain additional integrability conditions, the smoothness spaces related with a square-integrable group representation are defined by the decay of the associated voice transform. This technique is quite universal, and the classical smoothness spaces such as Besov and modulation spaces can be interpreted as coorbit spaces associated with the affine group and the Weyl-Heisenberg group, respectively. Moreover, the coorbit space theory provides a very general discretization technique which produces atomic decompositions and Banach frames for the coorbit spaces.

In [2], it has been clarified that the coorbit theory is indeed applicable to the full shearlet group, and in [3] the whole analysis has been generalized to arbitrary space dimensions. However, once these new smoothness spaces, the shearlet coorbit spaces, are established, some natural questions arise. How do these spaces really look like? Are there 'nice' sets of functions that are dense in these spaces? What are the relations to classical smoothness spaces such as Besov spaces? Do there exist embeddings into Besov spaces? And do there exist generalized versions of Sobolev embedding theorems for shearlet coorbit spaces? Moreover, can the associated trace spaces be identified? Concerning these questions, in [4, 5], the following results have been shown:

- for large classes of weights, variants of Sobolev embeddings exist;
- for natural subclasses which in a certain sense correspond to the 'cone adapted shearlets', there exist embeddings into (homogeneous) Besov spaces:
- for the same subclass, the traces onto hyperplanes can be identified with homogeneous Besov spaces or again with shearlet coorbit spaces.

Our approach heavily relies on atomic decomposition techniques. Recall that the coorbit space theory naturally gives rise to Banach frames, and therefore, by using the associated norm equivalences, all the tasks outlined above can be studied by means of weighted sequence norms of frame expansion coefficients. To make this approach really powerful, it is very convenient and sometimes even necessary to work with compactly supported building blocks. In the shearlet case, this is a nontrivial problem, since usually the analyzing shearlets are band-limited functions. In [4], it has been shown that indeed a compactly supported function with sufficient smoothness and enough vanishing moments can serve as an analyzing vector for shearlet coorbit spaces.

In this series of lectures, we will give an overview on all these recent developments.

References

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