UNDERSTANDING AND COMPARING SPARSITY PROPERTIES OF DIFFERENT FRAME CONSTRUCTIONS VIA DECOMPOSITION SPACES

In this short lecture series, I will present the construction of **decomposition spaces** and discuss how these can be used for understanding and comparing the *sparsity properties* of different frame constructions. In particular, the theory applies to **Gabor**, **wavelet** and **shearlet** frames.

Such sparsity properties are important for applications in image processing ranging from image *denoising* over *inpainting* to image *compression*; e.g., the JPEG 2000 image compression standard is based on wavelets. But sparsity is also crucial as a mathematical property, in particular for (infinite dimensional) **compressive sensing** and for understanding **nonlinear approximation** properties of certain function classes.

When students are introduced for the first time to Gabor, wavelet of shearlet frames, it is customary to draw pictures of the associated tilings of the Fourier domain. These pictures show the (essential) frequency supports of the different frame elements: For Gabor frames, this is a **uniform covering**; wavelet frames are associated with a family of dyadic annuli and shearlets are associated with a covering that divides the frequencies into two cones and each of these cones into dyadic sections which are then subdivided into wedge-like sets, cf. Figure 1. But in most cases, the precise connection between these *frequency coverings* and the properties of the associated frames is not made precise.

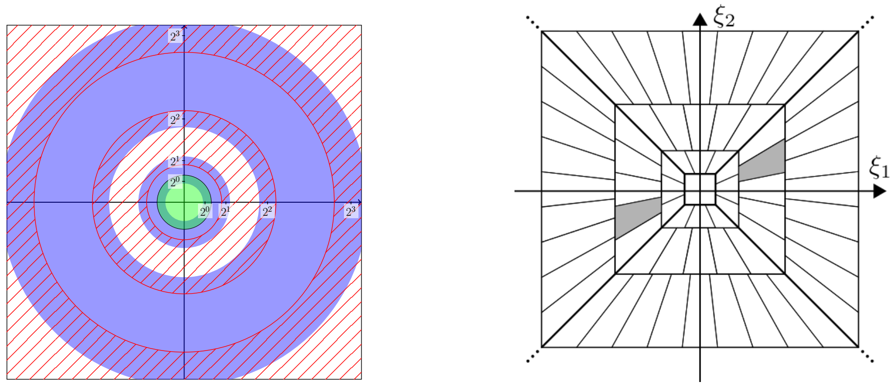

FIGURE 1. Left: The frequency tiling associated to a wavelet system. Right: The frequency tiling associated to a (cone-adapted) shearlet system. Right image courtesy of Martin Genzel.

This is where decomposition spaces come in: These start from an (almost) arbitrary covering $\mathcal{Q} = (Q_i)_{i\in I}$ of the frequency space $\mathbb{R}^d$ and use this covering to define an associated **smoothness space**. To compute the norm on this space, one first divides the given function/distribution g into its different **frequency parts** $g_i = \mathcal{F}^{-1}(\varphi_i \cdot \widehat{g})$, where $\Phi = (\varphi_i)_{i\in I}$ is a *partition of unity* subordinate to $\mathcal{Q}$. The individual parts g_i are measured in L^p and these individual contributions are aggregated using a certain weighted ℓ^q-norm. Formally, the norm is thus given by

$$\|g\|_{\mathcal{D}(\mathcal{Q},L^p,\ell^q_w)} = \left\|(w_i \cdot \|g_i\|_{L^p})_{i\in I}\right\|_{\ell^q} \in [0,\infty],$$

where $w = (w_i)_{i\in I}$ is a certain weight on the index set I of the covering $\mathcal{Q} = (Q_i)_{i\in I}$. The decomposition space $\mathcal{D}(\mathcal{Q}, L^p, \ell^q_w)$ itself is simply the set of all those distributions for which this norm is finite.

It is then intuitively plausible that the decomposition space describes sparsity with respect to frames whose *frequency localization* is similar to $\mathcal{Q}$. This intuition is formalized by the recent theory of **structured Banach frame decompositions of decomposition spaces**: Roughly speaking, it states that if a frame $\Psi = \left(\psi^{[i,k]}\right)_{i\in I,\,k\in\mathbb{Z}^d}$ is *compatible* with the covering $\mathcal{Q} = (Q_i)_{i\in I}$ in a certain sense, then the following are equivalent for a function/distribution f:

- f belongs to the decomposition space $\mathcal{D}\left(\mathcal{Q}, L^p, \ell^p_{w^{(p)}}\right)$ for a certain weight $w^{(p)} = w^{(p)}(\mathcal{Q})$.
- f is **analysis sparse** with respect to Ψ, i.e., the analysis coefficients $A_\Psi f = \left(\left\langle f, \psi^{[i,k]}\right\rangle\right)_{i\in I,\,k\in\mathbb{Z}^d}$ satisfy $A_\Psi f \in \ell^p$.
- f is **synthesis sparse** with respect to Ψ, i.e., $f = \sum_{i\in I,\,k\in\mathbb{Z}^d} c_k^{(i)} \cdot \psi^{[i,k]}$ for some sequence $c = (c_k^{(i)})_{i\in I,\,k\in\mathbb{Z}^d} \in \ell^p$.

In particular, if one chooses the uniform covering, the dyadic covering or the usual shearlet covering, then Gabor-frames, wavelet frames or shearlet frames are compatible with the respective covering, as long as the generators are sufficiently 'nice' and if the sampling density is fine enough. These 'nice' generators can be chosen to be compactly supported.

If time permits, we will also briefly discuss the **embedding theory** of decomposition spaces: Just by comparing two different coverings $\mathcal{Q}, \mathcal{P}$ of the frequency space, one can decide whether there is an embedding

$$\mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell^{q_1}_{w^{(1)}}\right) \hookrightarrow \mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell^{q_2}_{w^{(2)}}\right)$$

between the two associated decomposition spaces. In view of the theory from above, this allows one to decide whether sparsity with respect to one type of frame implies sparsity with respect to another type of frame, e.g. whether sparsity with respect to a Gabor frame implies sparsity with respect to a shearlet frame or vice versa.